NATIONAL MATHEMATICS MAGAZINE

(Formerly Mathematics News Letter)

Vol. XII BATON ROUGE, LA., FEBRUARY, 1938 No. 5

PUBLISHED BY LOUISIANA STATE UNIVERSITY

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Published by the Louisiana State University Press

Subscription, $1.50 Per Year in Advance. Single Copies, 20c.

Vol. XII BATON ROUGE, LA., FEBRUARY, 1938 No. 5

Published 8 Times Each Year by Louisiana State University. Vols. 1-8 Published as Mathematics News Letter.

All Business Communications should be addressed to the Editor and Manager P. O. Box 1322, Baton Rouge, La.

This Journal is dedicated to the following aims:

1. Through published standard papers on the culture aspects, humanism and history of mathematics to deepen and to widen public interest in its values.

2. To supply an additional medium for the publication of expository mathematical articles.

3. To promote more scientific methods of teaching mathematics.

4. To publish and to distribute to groups most interested high-class papers of research quality representing all mathematical fields.

The Economy of Symmetry

Every student of the Differential Calculus is introduced at an early stage to the methods of obtaining maximum and minimum values of functions. He is also taught to apply these methods to some geometric problems. A great opportunity is here given teachers to show the student the almost universal application of this knowledge. If manufacturers and contractors would engage a mathematical expert, as well as an expert chemist, or an expert efficiency man, they would gain great economy in various constructions. Nature is perfect, whether animate or inanimate. The basic forms of crystals are necessarily varied, in order to distinguish different minerals from each other. But each crystal form is such as to secure the greatest economic advantage.

Perhaps the best illustration of practical economy is shown by the bees, in the building of the wax cells for depositing their honey in the hives. These cells are hexagonal in cross section, with trihedral ends, which enable the cells in the hive to dove tail into each other perfectly and thus secure strong joints. But these ends are of strange dimensions.

The angles of the rhombi which form these trihedral ends, are startling to say the least. They are 109° 28′, and 70° 32′ respectively. The bees choose the hexagonal cross section, for their cells, and it is easy to show that the hexagon is the most economic polygon of the three which can fill a plane completely. These three are evidently the equilateral triangle, the square, and the regular hexagon.

No one would dream that the bees would always terminate their hexagonal prismatic cells by pyramidal ends with such peculiar face angles.

But they do this always, when undisturbed by obstructions. The Italian physicist, Maraldi, first measured the angles of these rhombus terminals. When the Swiss scientist, Rèaumur, learned of these angular dimensions, he suggested to the German mathematician, König, that perhaps these cells were the most economic possible. König solved the problem fairly accurately, but concluded that the angles of these rhombi should be 109° 30′ and 70° 30′ in order that the cells might contain the greatest amount of honey for a given amount of wax used in building them.

He therefore concluded that the bees were approximately right.

A century later, the Scotch mathematician, Maclaurin, and the French scientist, d'Huillier, solved the problem again, independently, and found that the bees were exactly right, and that König was slightly in error in his calculations. The economy gained by the bees is such that they can build 53 cells of this peculiar form, by using the same amount of wax as would be required to build 51 cells of right prismatic form, to contain the same volume. How the bees do this splendid engineering work, we do not know, but we call their faculty INSTINCT. The Creator of the universe and of all things in it, animate and inanimate, must have been, as Sir James Jeans asserts, a Pure Mathematician. To put such powers in animals and insects is humanly inconceivable.

Another very important problem employing the theory of maxima and minima is that of finding the relative dimensions of containers, such as cylinders or rectangular parallelopipeds, to contain the greatest possible volume, with a given constant surface, or a minimum surface with a constant volume.

The student should be shown that if the solid is closed, it will be most economic if the altitude equals the base diameter, if cylindrical. If the solid is a rectangular piped, it will be most economic if it is a cube, provided in each case the solid is closed. If open at top, it will be found that the altitude should be half as great as the diameter of the cylinder, and half as great as each dimension of the square base.

In fact, the student should be taught that economy in closed containers will always be obtained when the dimensions are as nearly equal as possible, when there is perfect freedom in choosing the relative dimensions. When open, one dimension will be found to make the economy perfect, if it is half as great as the other dimension.

Of course we know that in practice, one is not usually allowed perfect freedom in the choice of dimensions, since at least one dimension is necessarily limited to the requirement of the problem. Let us illustrate by a familiar example. We shall consider the dimensions of a sardine can. Let us assume that the sardines are about five inches long.

This limits the length of the can to five inches. The can is usually flat, with dimensions about four by one inch.

The slight rounding of the corners does make a slight loss in economy, but it helps prevent chafing of the hands in handling great numbers of the cans. The volume of this can is evidently $5 \times 4 \times 1$ *or 20 cubic inches, while the total surface, that is, the amount of tin required for construction*

is 58 square inches. If now we should make the can five by two by two inches the volume would be the same, that is, the new can would hold the same quantity of sardines. But the surface will be only 48 square inches, thus saving ten square inches, or more than 20 per cent of tin in the construction. In making millions of cans, this makes a large saving. In fact the saving is from one-half to two-thirds of a cent for each can.

Some may object to the square form, because the flat form of can will exhibit the sardines to better advantage. But it surely is not desirable to present sardines to banqueters in tin cans.

A rectangular wooden box, to contain 8 cubic feet of space, may be made, if unrestricted, in the form of a cube, two feet on each edge, or four four feet long, with ends two by one, or eight feet long with square ends, one by one, or in many other forms. The cubic form will be found to secure a large saving in surface material used in its construction.

We thus see that containers may be made extravagant or economical, by carefully adjusting the dimensions which are free to be adjusted as we wish. We may save greatly in the manufacture of many sorts of containing vessels, boxes or cans. The principles here illustrated, are perfectly general. The teacher should instill in the minds of his students the practical advantage resulting from economic construction.

Many mathematicians are interested only in teaching pure mathematics, but if they are instructing embryonic engineers, they will accomplish a great deal for their students if they would impress upon them the fact that every object has its own most perfect dimensions for economy, as well as for efficiency.

JAMES McGIFFERT.

An Ellipsograph

By ROBERT C. YATES
University of Maryland

In an article *On the Geometry of Clocks*, The American Mathematical Monthly, Vol. 37 (1930), p. 368, J. M. Feld has given a method of constructing an ellipse by means of vectors rotating about a fixed point in opposite directions. The present note is concerned with the the construction of a linkage, embodying Feld's idea, which will mechanically describe the ellipse and at the same time produce straight line motion. It is to be admitted that the linkage here presented, containing seven bars, is not the simplest known but it is apparently novel.

Consider the *map* equation:

$$\boxed{z = \rho_1 t + \rho_2/t} \tag{1}$$

where z is the complex variable $x+iy$, ρ_1 and ρ_2 are real, and $t=e^{i\Theta}$. As Θ varies, the point t traverses the unit circle and, for fixed values of ρ_i, z describes an ellipse. The rectangular equation of the ellipse may be had by writing for t its equivalent expression, $\cos\Theta + i\cdot\sin\Theta$:

$$x+i\cdot y = \rho_1(\cos\Theta + i\cdot\sin\Theta) + \rho_2(\cos\Theta - i\cdot\sin\Theta)$$

and equating reals and imaginaries:

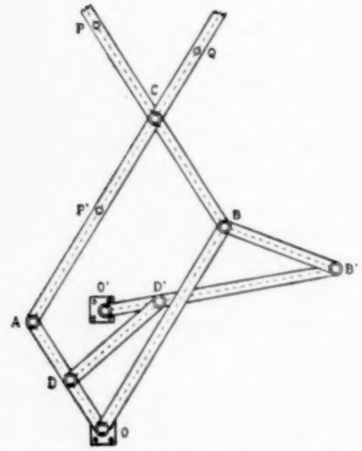
$$x = (\rho_1+\rho_2)\cos\Theta, \qquad y = (\rho_1-\rho_2)\sin\Theta.$$

Thus (1) is equivalent to:

$$\boxed{x^2/(\rho_1+\rho_2)^2 + y^2/(\rho_1-\rho_2)^2 = 1} \tag{2}$$

Now Equation (1) expresses z as the sum of two vectors, that is, the diagonal of the parallelogram with sides ρ_1 and ρ_2, the first side rotating about the origin in one direction and the second in the *opposite* direction *at the same rate*.

In the accompanying figure, $OACBO$ is the parallelogram in question, composed of links or jointed rods with $OA=\rho_1$, $OB=\rho_2$. In order to produce elliptic motion we must arrange matters so that OA and OB make equal angles at all times with some direction OO'. This



may be accomplished by attaching two *contraparallelograms*, $OO'BB'$ and $OO'DD'$, as shown.

We take	$OO' = BB'$,	$OB = O'B'$
so that	angle $O'OB$ = angle $O'B'B$;	
and	$OD = O'D'$,	$DD' = OO'$
so that	angle DOO' = angle $DD'O'$.	

We require that angle DOO' = angle $O'OB$ so that the two contraparallelograms are similar. It follows that

$$OD/OO' = OO'/OB$$

or

$$(OO')^2 = (BB')^2 = (DD')^2 = (OD)(OB) = (O'D'(O'B').$$

Convenient lengths to be taken for the construction of the apparatus are as follows:

$$OO' = BB' = DD' = 6;\ OD = O'D' = 4;\ OB = O'B' = AC = 9.$$

The points O and O' are, of course, attached to a base plane.

It is obvious that either ρ_1 or ρ_2 (or both) may be altered to produce ellipses of different sizes. Thus *every* point, P, of BC, or Q of AC, describes an ellipse.

A case of special interest is that in which the parallelogram is a rhombus and $\rho_1 = \rho_2$.* The ellipse degenerates into a straight line

*It is not at all necessary here to move the bar AC. The points P and Q describe lines if $PB = OB$ and $AQ = OA$.

and we have another instance* of the conversion of circular into linear motion.

It is to be noticed that the partial linkage $OABCP$ formed of a parallelogram with an extended side is nothing more than an ordinary pantograph. If P and P' are collinear with O then the locus of either point is an enlargement, or reduction, of the locus of the other.

*See articles by Sylvester, Cayley, Kempe, Roberts, Hart, etc., in the British journals from 1870-1880, and in particular, *How to Draw a Straight Line*, (1877) by A. B. Kempe.

Expansions Involving Differential Equations in Which the Coefficient of a Parameter Changes Sign*

By CHESTER C. CAMP
University of Nebraska

1. *Introduction.* The boundary value problems considered in this paper have to do with a differential equation of the first order

(1) $$X'_1+[\lambda a(x)+b(x)]X_1=0$$

or with a system of the first order

(2) $$X'_j+\sum_{k=1}^{p}\lambda_k a_{jk}(x_j)X_j=0, \qquad j=1,2,\cdots,p$$

It is surprising that so little has been written on the first order as compared with the second order differential equation.† On first thought one might say that if the second order case has been adequately treated then the case of order one would be trivial. Such is decidedly not the case except when the coefficient $a(x)$ of the parameter λ does not change sign. If, however, this coefficient is allowed to change sign or otherwise vanish as in this paper there is a tremendous difference in the behavior of the expansions for the lower order as compared with those for the higher order.

Stone‡ treats the equation $u'+\lambda pu=0$ with the boundary conditions $u(0)-u(1)=0$ or $u(0)+u(1)=0$. However, instead of expanding $f(x)$ in the ordinary way he considers the simultaneous expansion of two related functions called P-average functions in terms of two different subsets of characteristic solutions. $p(x)$ is so restricted that with no loss in generality he might have assumed it continuous with a finite number of changes in sign and that

$$\int_0^1 p(x)dx=1,$$

except that $p(x)=0$ only for a set of points of zero measure.

*Presented to the Society, November 28, 1936.

†This reversal of order of treatment has doubtless been due to the important applications to Physics.

‡M. H. Stone, *An Unusual Type of Expansion Problem*, Transactions of the American Mathematical Society, Vol. 26, (1924), p. 335.

In the present paper the coefficient of the parameter is allowed to vanish identically throughout a countable number of sub-intervals as well as to change sign a like number of times.* Moreover the method of proof of convergence is different. The contour integral method is used while Stone used a method of comparison to Fourier series. Instead of Stone's somewhat artificial related functions the author expands $f(x)$ directly and determines to what (in terms of f) the series converges. In section three an interesting special case is treated in some detail. Later it is indicated how the theory can be extended to the system (2) as well as to a more general system of boundary conditions.

2. *Case of $p=1$.* By the transformation $X_1 = X/exp \int b(x)dx$ equation (1) may be reduced to the form

$$X' + \lambda a(x) X = 0. \tag{3}$$

We consider this with the boundary condition

$$X(-1) = X(1). \tag{4}$$

Here $X = e^{-\lambda A(x)}$ where $A(x) = \int a(x)dx$, and the solution of the adjoint system $-Y' + \lambda a(x) Y = 0$, $Y(-1) = Y(1)$ is $Y = e^{\lambda A(x)}$.

The principal parameter values are $\lambda_n = n\pi i/A_1$, $n = 0, \pm 1, \pm 2, \cdots$, where

$$A_1 = \tfrac{1}{2}\int_{-1}^{1} a(x)dx$$

must be assumed not to vanish. The Green's function is

$$G(x,s,\lambda) = \begin{cases} exp \displaystyle\int_x^s \lambda a(t)dt/(1 - e^{-2\lambda A_1}), & s < x; \\ exp \displaystyle\int_x^s \lambda a(t)dt/(e^{2\lambda A_1} - 1), & s > x. \end{cases} \tag{5}$$

*For the second order case in which the coefficient changes sign see: Lichtenstein *Zur Analysis der unendlichvielen Variabeln*, Rendiconti del Circolo Matematico di Palermo, Vol. 38, (1914), pp. 113-166; Anna Pell Wheeler, *Linear Ordinary Self-Adjoint Differential Equations of the Second Order*, American Journal, Vol. 49, (1927), p. 309; and Langer, *The Boundary Problem Associated with a Differential Equation in Which the Coefficient of the Parameter Changes Sign*, Transactions of the American Mathematical Society, Vol. 31, (1929), pp. 1-24.

The formal expansion is

$$f(x)=\sum_{n=-\infty}^{\infty} C_n e^{-\lambda_n A(x)} \text{ where} \tag{6}$$

$$C_n=\int_{-1}^{1} f(s)a(s)exp\lambda_n A(s)ds/2A_1, \quad n=0,\ \pm 1,\ \pm 2,\cdots$$

This is represented, as is well known, by the limit of

$$\frac{1}{2\pi i}\int_{\Gamma}\int_{-1}^{1} f(s)a(s)\,G(x,s,\lambda)ds\,d\lambda$$

as the circular contour Γ with center at $\lambda=0$ znd radius $|\lambda|=(n+\frac{1}{2})\pi/A_1$ expands beyond limit. Call this limit $F(x)$. Then one has for n large

$$F(x)\sim\frac{1}{2\pi i}\int_{\Gamma}\int_{-1}^{x} f(s)a(s)exp\lambda[A(s)-A(x)]ds\,d\lambda/(1-e^{-2\lambda A_1})$$

$$+\frac{1}{2\pi i}\int_{\Gamma}\int_{x}^{1} f(s)a(s)exp\lambda[A(s)-A(x)]ds\,d\lambda/(e^{2\lambda A_1}-1)$$

or, after integration by parts, $F(x)\sim$

$$\frac{1}{2\pi i}\int_{\Gamma}\Bigg\{f(1)exp\lambda[A(1)-A(x)]-f(x+0)$$

$$-\int_{x}^{1} f'(s)exp\int_{x}^{s}\lambda a(t)dt\,ds\Bigg\}d\lambda/\lambda(e^{2\lambda A_1}-1)$$

$$+\frac{1}{2\pi i}\int_{\Gamma}\Bigg\{f(x-0)-f(-1)exp\lambda[A(-1)-A(x)]$$

$$-\int_{-1}^{x} f'(s)exp\int_{x}^{s}\lambda a(t)dt\,ds\Bigg\}d\lambda/\lambda(1-e^{-2\lambda A_1}).$$

In order to evaluate this limit one needs the following

Lemma I:

$$\lim_{|z|\to\infty}\int_C \frac{e^{hz}dz}{2\pi iz(e^z-1)}=\lim_{|z|\to\infty}\int_C \frac{e^{-hz}dz}{2\pi iz(e^{-z}-1)}=h-\tfrac{1}{2},$$

when h is any integer (positive, negative, or zero); and $h-\theta$ when $h=h'+\theta$, $0<\theta<1$, and h' is an integer. Here C is a circular contour with center at $z=0$, which passes half way between poles.

The proof consists of first dividing by the binomial divisor until the coefficient of z in the exponent of the numerator is less than one numerically and then applying lemmas previously known.*

In case $f(x)$ has one or more discontinuities it is necessary to break the integrals in the above integrations by parts at these points. If f consists of horizontal segments the integrals containing $f'(s)$ disappear leaving $\frac{1}{2}f(x+0)+\frac{1}{2}f(x-0)+$constant multiples of $f(1)-f(-1)$ and $f(\xi_i+0)-f(\xi_i-0)$ where $x=\xi_i$ is a point of discontinuity of $f(x)$.

To evaluate the integrals containing $f'(s)$ in general it is expedient to consider Fundamental Subintervals for $A(x)$. Break up the interval from -1 to 1 using the following points of division: points a_i, at which $A(x)-A(-1)=0$; points a_{ki} at which $A(x)-A(-1)=2kA_1$, $k=\pm 1, \pm 2, \cdots$; points b_i where $a(x)=0$ or extremities of segments throughout which $a(x)\equiv 0$; other points d_i at which $A(x)$ has an extremum; points b_{ki} for which $A(x)-A(b_i)=2kA_1$; and d_{ki} at which $A(x)-A(d_i)=2kA_1$.

Let x be in one of these subintervals. In order to evaluate the contour integral completely it is necessary to consider points ζ_i for which $A(x)-A(\zeta_i)=0$ and points ζ_{ki} at which

$$A(\zeta_{ki})-A(x)=2kA_1, \quad k=\pm 1, \pm 2, \cdots \tag{8}$$

Let f be made up of a finite number of pieces, each real, continuous, and possessing a continuous derivative. Then if one reverses the order of integration in the terms containing $f'(s)$ it is possible to determine completely the limit $F(x)$ for any particular x. The form obtained for x in one of the fundamental subintervals will hold in general for x within that subinterval. Points of division should be treated separately. From these considerations one may state the following

Theorem I. Let f consist of a finite number of pieces, each real, continuous, and possessing a continuous derivative; let $a(x)$ be real integrable, $-1\leq x\leq 1$, but

$$\int_{-1}^{1} a(s)ds\neq 0.$$

*Birkhoff and Langer, *Boundary Problems and Developments Associated with a System of Ordinary Linear Differential Equations of the First Order*, Proceedings of the American Academy, Vol. 58, (1923), pp. 112-114.

The expansion (6) will always converge but not always to the mean value of f. It will in general have additional terms including multiples of the saltus of f at its points of discontinuity, of $f(1)-f(-1)$, and of $f(\zeta_i)$, $f(\zeta_{ki})$ where ζ_i and ζ_{ki} depend on x as defined above.

3. *Application.* An expansion which exhibits striking behavior occurs when $a(x)=1+2x$. Let $f(x)$ have a finite discontinuity at $x=\frac{1}{2}$. Here one finds that in one subinterval the expansion converges to the mean value of $f(x)$ without additional terms. The points of division are $x=-\frac{1}{2},0,2^{\frac{1}{2}}-\frac{1}{2}$. For the first two subintervals $\zeta_1=-1-x$, $\zeta_{11}=(x^2+x+9/4)^2-\frac{1}{2}$. For the last one $\zeta_{-11}=-\frac{1}{2}-(x^2+x-7/4)^{\frac{1}{2}}$, $\zeta_{-12}=-\frac{1}{2}+(x^2+x-7/4)^{\frac{1}{2}}$. If we define $f(x)=f_1(x)$, $-1\leqq x<\frac{1}{2}$, $f(x)=f_2(x)$, $\frac{1}{2}<x\leqq 1$, then the series $F(x)$ converges as follows:

$$
\begin{aligned}
&\text{for } -1<x\leqq -\tfrac{1}{2},\ F(x)=f_1(\zeta_1)-f_1(x)+f_2(\zeta_{11});\\
&\text{for } -\tfrac{1}{2}\leqq x<0,\ F(x)=f_1(x)-f_1(\zeta_1)+f_2(\zeta_{11});\\
&\text{for } 0<x\leqq 2^{\frac{1}{2}}-\tfrac{1}{2},\ F(x)=\tfrac{1}{2}f(x+0)+\tfrac{1}{2}f(x-0);\\
&\text{for } 2^{\frac{1}{2}}-\tfrac{1}{2}\leqq x<1,\ F(x)=f_2(x)+f_1(\zeta_{-11})-f_1(\zeta_{-12}).
\end{aligned} \tag{9}
$$

The values at $-1,0,1$ are not correctly obtained here by simply taking the limits approached. With no difficulty one may verify

Theorem II: Within a subinterval for x in which $A(x)-A(s)=2hA_1$, $-1\leqq s<x$, and $A(s)-A(x)=2h'A_1$, $x<s\leqq 1$, where $0<h<1$, $0<h'<1$, the series (6) denoted by $F(x)$ will converge to the mean value of $f(x)$.

The convergence of (6) as given in (9) has been verified for several choices of f_1, f_2 by actually finding C_n, combining terms, and summing by means of the formulas

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{\sin nx}{n}=\frac{\pi(1+2k)-x}{2},\ 2k\pi<x<2(k+1)\pi,\ k=0,\pm1,\pm2,\cdots\\
&\sum_{n=1}^{\infty}\frac{\cos n\pi x}{n^2\pi^2}=\frac{3x^2-6x+2}{12},\quad 0\leqq x\leqq 1.
\end{aligned} \tag{10}
$$

This can be done even when the integrals needed in determining C_n cannot be carried out, by summing under the integral sign. It is of interest to note that the series (6) is a generalization of ordinary Fourier series.

4. *Case of* $p \geqq 2$. Consider the system (2) or a more general system which may be reduced to (2) by transformations of the form

$$u_j = X_j / exp \int b_j(x_j) dx_j, \quad j = 1, 2, \cdots, p. \tag{11}$$

Let the boundary conditions be such that by (11) they become

$$X_j(a) = g_j X_j(b), \quad (g_j > 0), \quad j = 1, 2, \cdots, p. \tag{12}$$

If one removes the restriction that $a_{jk}(x_j)$ maintains its sign, $a \leqq x_j \leqq b$, but assumes

$$\int_a^b a_{jk}(x_j) dx_j \neq 0,$$

most of the work will proceed as in previous treatments.* One needs the following extension of Lemma I, namely

Lemma II:

$$\begin{aligned} \lim_{|z| \to \infty} \int_\Gamma \frac{e^{-hz} dz}{2\pi i z(1 - g e^{-z})} = & \sum_{s=0}^{[-h]} g^s, \ h < 0 \text{ and non-integral;} \\ & - \sum_{s=1}^{[h]} g^{-s}, \ h > 0 \text{ and non-integral;} \\ & \tfrac{1}{2}, \ h = 0; \\ & \sum_{s=0}^{-h} g^s - \tfrac{1}{2} g^{-h}, \ h < 0 \text{ and ntegral;} \\ & - \sum_{s=1}^{h} g^{-s} + \tfrac{1}{2} g^{-h}, \ h > 0 \text{ and integral.} \end{aligned} \tag{13}$$

Here Γ is a circular contour with center at $z = 0$ bounded uniformly away from the poles. The method of proof is as above.

By the use of this lemma one sees that in general the multiple series will converge to the mean value of $f(x_1, x_2, \cdots, x_p)$ plus extra terms involving constant multiples of values of f at various points of the p-fold region. One may however clearly state

*Cf. among others: Camp, *An Expansion Involving p Inseparable Parameters Associated with a Partial Differential Equation*, American Journal of Mathematics, Vol. 50, (1928), pp. 259-268.

Theorem III: Within a subregion for $x_1,x_2,\cdots,x_p$ in which $A_{jk}(x_j)-A_{jk}(s_j)=h_j(b-a)A_{jk}$, $a\leqq s_j<x_j$, where $0<h_j<1$, and $A_{jk}(s_j)-A_{jk}(x_j)=h_j(b-a)A_{jk}$, $x_j<s_j\leqq b$, where $0<h'_j<1$, the usual multiple series expansion

$$(14)\qquad f(x_1,x_2,\cdots,x_p)=\sum_{m_j=-\infty}^{\infty} C_{m_j}\prod_1^p X_j^*$$

analogous to that given by the author* will still converge to the so-called mean value of f.

*Loc. cit.

Humanism and History of Mathematics

Edited by
G. WALDO DUNNINGTON

Ueber die Quadraturen des Artus de Lionne

VON JOS. E. HOFMANN
in Nördlingen

1. *Einleitung.* ARTUS DE LIONNE (1583-1663) war kein Mathematiker von Fach und schon zu seiner Zeit kein Stern erster Grösse am mathematischen Himmel. In den Werken über die Geschichte der Mathematik wird er nur ganz flüchtig erwähnt als Verfasser einer Schrift über die Quadratur kreisbogenförmig begrenzter Flächenstücke. Es schien kaum der Mühe wert, die Arbeit LIONNES näher anzusehen. Die genauere Untersuchung zeigte aber, dass die Schrift des LIONNE keineswegs so bedeutungslos ist, als es den Anschein hatte. Es handelt sich um eine Jugendarbeit, die viele Jahre nach ihrer Abfassung von dem Jesuitenpater VINCENT LÉOTAUD (1596-1672) herausgegeben wurde.[1] Sie bildet den ersten Teil einer weitschichtigen Streitschrift LÉOTAUDS gegen eine misslungene Kreisquadratur seines Ordensgenossen GREGORIUS A SANCTO VINCENTIO (1584-1667), die schon nach wenigen Jahrzehnten in Vergessenheit geriet. Was von LIONNE selbst stammt und was vielleicht von LÉOTAUD hinzugefügt worden ist, lässt sich schwer entscheiden. Auch über die Quellen, aus denen LIONNE geschöpt haben kann, ist wenig Klarheit zu gewinnen. Im Text werden ausser der verbreiteten *Geometria practica* des Jesuiten CHRISTOPH CLAVIUS (1537-1612)[2] noch die *Elementa tetragonismica....* des Jesuiten ANTOINE DE LALOUBÈRE (1600-64)[3] erwähnt, nicht aber das berühmte *Variorum de rebus mathematicis Responsorum liber*

[1] V. LÉOTAUD: *Examen circuli quadraturae*...Lugduni 1654. Pars I: ARTUS DE LIONNE. *Amoenior curvilineorum contemplatio.* 4, (13)+116 p., 35 Figuren im Text. Die Abbildungen sind recht mässige Holzschnitte und nicht immer fehlerfrei.

[2] Romae 1604, Moguntiae 1606. Welche der beiden Ausgaben benutzt wurde, lasst sich aus der Art des Zitates nicht entscheiden.

[3] Tolosae 1651. Das Zitat scheint eine ganz nebensächliche Einschiebung des LÉOTAUD zu sein.

VIII des FRANÇOIS VIÈTE (1540-1603)[4], das von LIONNE zweifelsohne benutzt wurde. Da sich LIONNE nach dem frühen Tod seiner Gattin (1612) ganz von der Mathematik zurückzog, ist der erste Entwurf vor dieses Jahr anzusetzen.

Die Schrift des LIONNE beginnt mit einer kurzen Uebersicht, in der die 53 propositiones zu grösseren Einheiten zusammengefasst werden. Gewöhnlich finden sich längere Einschiebsel in Form von Zusätzen und Bemerkungen am Schluss mehrerer zusammengehöriger prop. Viele dienen als vorbereitende Sätze dem logischen Aufbau des Ganzen, waren aber schon damals Allgemeingut der Mathematiker und sind daher für uns bedeutungslos. Wieder andere sind nur dadurch in das Werkchen hineingekommen, dass LIONNE jegliches Rechenzeichen vermeidet und daher vieles umständlich und langweilig in Worten ausdrücken muss, was wir heute durch eine Gleichung in ein paar Zeilen hinschreiben können.

2. *Die Teile des Viertelmöndchens.* In den ersten 8 prop. untersucht LIONNE das quadrierbare Viertelmöndchen[5] des HIPPOKRATES VON CHIOS (etwa 440 v. Chr.)[6], das entsteht, wenn man von einem Halbkreis ABC (Abb. 1) das Segment ACD eines Viertelkreises über Sehne AC wegnimmt. Seine Fläche ist gleich dem rechtwinkliggleichschenkligen Dreieck AFC unter der Grundlinie AC. Nun folgen Hilfssätze über ähnliche Kreisteile (prop. 9-13) und die reizende prop. 14, mit der wir uns etwas eingehender befassen müssen: *Das Viertelmöndchen durch eine Gerade in gegebenem Verhältnis zu teilen,* Er will die Mondfläche so teilen, wie die Strecke RS durch T geteilt wird. Zu diesem Zweck teilt er AC durch G so, wie RS durch T geteilt wird, legt dann das Lot GH zu AC und zieht HF. Nun verhalten sich die gemischtlinigen Mondstücke HCI, HAI wie ST zu TR.

[4] Turonis 1593. VIÈTE behandelt dort im 11. Abschnitt zwei quadrierbare Kreismöndchen (prop. 2-5, fol. 18a-19a), die in dieser Form neu waren. Beide treten auch bei LIONNE auf (prop. 19 26). Ganz offenkundig strebt LIONNE darnach, das Verfahren des VIÈTE zu verbessern, was ihm auch gewissermassen gelungen ist. Die Darlegungen VIÈTES sind in den *Opera Mathematica* ed. FR. VAN SCHOOTEN: Leiden 1646, p. 378-80 wieder abgedruckt. Dort wird alles durch die Einführung der CARTESischen Schreibweise und Zeichensprache viel übersichtlicher. LIONNE hängt noch von der alten Ausgabe, ab; er vermeidet die Buchstabenrechnung völlig. Die scheint darauf hinzudeuten, dass LÉOTAUD den ursprünglichen Text LIONNES ziemlich unverändert herausgegeben hat.

[5] Der von mir zur Abkürzung verwendete Ausdruck *Viertelmöndchen = luna quadrantalis* tritt noch nicht bei LIONNE, sondern erst im 18. Jahrhundert auf.

[6] Das fragliche HIPPOKRATES-Bruchstrück ist uns bekannt aus dem ARISTOTLES-Kommentar des SIMPLIKIOS (etwa 520). Man sehe hierüber F. RUDIO: *Der Bericht des Simplicius über die Quadraturen des Antiphon und des Hippokrates.* Urkunden z. Geschichte d. Math. *1*, Leipzig 1907. Abschliessendes bringt O. BECKER: *Zur Textgestaltung des eudemischen Berichtes über die Quadratur der Möndchen durch Hippokrates von Chios.* Quellen u. Studien z. Geschichte d. Math. B *3*, 1936, S. 411-19. Ueber die Weiterentwicklung der quadrierbaren Möndchen findet man vieles bei J. E. HOFMANN-H. WIELEITNER: *Zur Geschichte der quadrierbaren Kreismonde.* Programm Neues Realgymnasium München, 1934.

Der Beweis ist kennzeichnend für die Denkweise LIONNES. Daher will ich ihn fast unverkürzt wiedergeben. Werden *EH* und *GF* gezogen, so ist Sektor *EHC* gleich Sektor *FIC*.[7] Jetzt wird zuerst von jedem Sektor das gemischtlinige Dreieck *KEC* weggenommen und dann zu jedem das gemischtlinige Dreieck *IHK* hinzugefügt. Dann ist das gemischtlinige Dreieck *HCI* gleich dem Dreieck *ECF*, zusammengenommen mit dem Dreieck *EHF* odoer mit dem ihm gleichen Dreieck *EGF*[8], also insgesamt gleich dem Dreieck *GFC*.

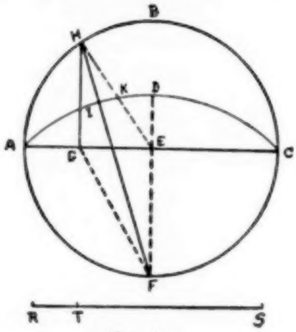


Fig. 1

Ausserdem ist das Möndchen gleich dem Dreieck *ACF*; folglich sind auch die Reste gleich, nämlich das gemischtlinige Dreieck *HAI* und das Dreieck *AFG*. Wie sich also das gemischtlinige Dreieck *HCI* verhält zu *HAI*, so verhalten sich auch die Dreiecke *GFC* und *GFA*, das heisst die Strecken *CG* und *CA* oder *ST* zu *TR*, wie es sein soll.

In einem zweiten Beweis wird durch eine dem ersten Verfahren entsprechende Schlussweise direkt gezeigt, dass das gemischtlinige Dreieck *HAI* und das Dreieck *GFA* flächengleich sind. Das brauche ich nicht mehr vorzuführen.

Diese Teilung des Viertelmöndchens gehört bestimmt zu den geistreichsten elementargeometrischen Miniaturen, die wir besitzen. Daher muss uns die Gleichgültigkeit überraschen, mit der sie von den Zeitgenossen totgeschwiegen wurde. Wir würden heute keineswegs die versteckte Teilungsaufgabe an die Spitze stellen, sondern den Satz von der Gleichheit des gemischtlinigen Dreiecks *CHI* und des Dreiecks *GFC*. So wird es auch vom Freiherrn EHRENFRIED WALTHER VON TSCHIRNHAUS (1651-1708) gemacht[9], dem man

[7] Denn die Zentriwinkel der Sektoren verhalten sich wie 2:1, ihre Halbmesser wie $1:\sqrt{2}$. LIONNE stützt sich auf einen seiner früheren Hilfssätze.

[8] Die Dreiecke haben die gleiche Grundlinie *EF*; ihre Spitzen *G*, *H* liegen auf der gleichen Parallelen zur Grundlinie.

[9] D[omini] T[schirnhusii] *Additamentum ad Methodum quadrandi curvilineas figuras aut earum impossibilitatem demonstrandi per finitam seriem.* Acta Eruditorum 1687, p. 524-27.

bisher den Satz zuschrieb. TSCHIRNHAUS will ihn selbständig entdeckt haben[10]; den Beweis übergeht er "wegen seiner Leichtigkeit."

3. *Die Teile des Kreisbeils.* In prop. 15 bildet LIONNE ein Kreisbogendreieck *ADCSFR*—ich nenne es abkürzend ein Kreisbeil—das mit dem Viertelmöndchen flächen—und umfangsgleich ist. Wie es erzeugt wird, geht unmittelbar aus Abb. 2 hervor.[11] Nun wird

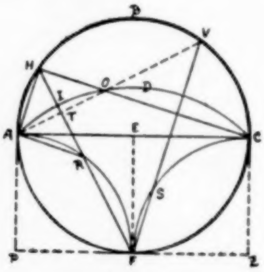


Fig. 2

(prop. 16) eine beliebige Gerade *RIH* durch *F* gelegt und behauptet, dass die Teile *HAI* und *AIR* des Möndchens und des Kreisbeils gleich sind; ebenso auch die Teile *HCI* und *ICSFR*. Zum Beweis fällt LIONNE das Lot *AT* auf *FH*, benutzt die Gleichheit der Segmente *AH*, *AR* mit dem Halbsegment *AIT* und zeigt, dass die Flächenteile *HAI* und *AIR* flächengleich sind mit den kongruenten gleichschenklig-rechtwinkligen Dreiecken *HAT* und *ATR*. Nun wird (prop. 17) *AT* bis zum Schnitt *V* verlängert und *VF* gezogen. Dann ergibt sich, dass das rechtwinklige Dreieck *FTV* gleich dem Teil *HCI* des Möndchens und gleich dem Teil *ICSFR* des Beils ist. Schliesslich sollen wir noch *HC* ziehen (prop. 18): dieses Gerade schneidet *AV* in einem Punkt 0 des Viertelbogens *AIDC* und es wird $AH=HO$.

Der Satz, dass das gemischtlinige Dreieck *HAI* gleich dem Dreieck *HAT* ist, wurde erstmals wiedergefunden von dem englischen Arzt JOHN PERKS (1699).[12] Dass das Stück *AIR* des Kreisbeils gleich dem Dreieck *ATR* ist, entdeckte später auch der Marquis GUILLAUME FRANÇOIS DE L'HÔPITAL (1661-1704), der mehrere schöne

[10] Dies trifft vermutlich nicht zu; vielmehr ist TSCHIRNHAUS sehr wahrscheinlich abhängig von LIONNE, auf dessen Schrift er recht wohl durch G. W. LEIBNIZ hingewiesen worden sein kann. Ueber die Gründe für diese Annahme sehe man HOFMANN-WIELEITNER[6] p. 41-43 nach.

[11] Im Original fehlt der (im Text auftretende) Punkt I; ausserdem stört an der Figur, dass *O* fast auf *D* fällt.

[12] Englisch (mit ellenlangem Titel) abgedruckt in den Philosophical Transactions *21*, 1699 (gedruckt 1700), p. 411-18. In lateinischer Uebersetzung in den Acta Eruditorum 1700, p. 306-12. Ueber die weitere Entwicklung siehe HOFMANN-WIELEITNER[6] p. 45-47.

Erweiterungen hinzufügte.[13] An der Selbständigkeit dieser beiden Untersuchungen besteht kein Zweifel.

4. *Die Kreismöndchen am allgemeinen rechtwinkligen Dreieck.* Wir übergehen die nun folgenden prop. 19-26, in denen sich LIONNE mit den von VIÈTE behandelten Kreismöndchen beschäftigt.[4] Hingegen müssen wir prop. 27 wieder vornehmen. LIONNE geht hier (Abb. 3) aus von einem allgemeinen rechtwinkligen Dreieck *ABC* und legt über die Seiten die Halbkreise. So entstehen zwei Möndchen, die zusammen ebenso gross sind wie das rechtwinklige Dreieck. Der Beweis ist kurz und bündig: Nach dem Lehrsatz des PYTHAGORAS ist das Quadrat über *AB* gleich der Summe der Quadrate über *AC* und

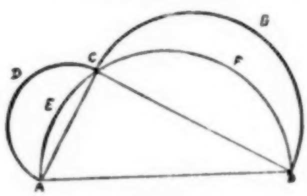


Fig. 3

BC. Kreise und Halbkreise verhalten sich wie die Quadrate über ihren Durchmessern. Also ist Halbkreis *ACB* gleich der Summe der Halbkreise *CDA*, *CGB*. Nach Wegnahme der gemeinsamen Segmente *CEA*, *CFB* ist Dreieck *ACB* gleich der Summe der Möndchen *CDAE* und *CGBF*. *Ist Dreieck ABC gleichschenklig*, fügt LIONNE bei, *so entstehen zwei gleiche Viertelmöndchen und jedes ist gleich der halben Dreiecksfläche. Ist aber das Dreieck nicht gleichschenklig, so ist die Teilung des Dreiecks im Verhältnis der beiden Einzelmöndchen nicht leichter als die Quadratur des Kreises.*

Dieser Satz von den Kreismöndchen am allgemeinen rechtwinkligen Dreieck erscheint zum erstenmal bei dem Araber IBN ALHAITAM (965-1039).[14] Sein Beweis beruht auf dem gleichen Grundgedanken, ist aber in der Ausführung viel umständlicher. Bisher glaubte man, der Satz trete im Abendland zum erstenmal in den *Élémens de Géométrie* des Jesuitenpaters GASTON PARDIES (1637-73) auf.[15] In Wahrheit ist LIONNE die unmittelbare Quelle für PARDIES, der

[13] *La quadrature absolue d'une infinité de portions moyennes tant de la Lunule d'Hippocrate de Chio, que d'une autre d'une nouvelle espace.* Mémoire Ac. sc. Paris, Amsterdam 1701 (gedruckt 1707), p. 22-26; Abbildungen zwischen p. 50 und 51. Die dazu gehörige Uebersicht steht in der Histoire Ac. sc. Paris des gleichen Jahres, p. 98-100.

[14] H. SUTER: *Die Kreisquadratur des Ibn el Haitam.* Z. Math. Phys. *44*, 1899, hist.-liter. Abt., p. 33-47.

[15] Paris 1671, Livre IV, §63 = p. 79-80 der lateinischen Ausgabe: Jena 1684 (nach der dritten französischen Auflage gefertigt).

übrigens seine Vorlage nur sehr flüchtig angesehen hat.[16] PARDIES nennt seinen Vorgänger nicht, aber er verrät seine Abhängigkeit von ihm deutlich durch Uebernahme der Beifügung LIONNES, die nichts mehr mit dem eigentlichen Satz zu tun hat.

5. *Der weitere Inhalt der Schrift.* Die noch verbleibenden Sätze sind ziemlich unbedeutend. In prop. 28 werden zwei einander von innen berührende Kreise im Flächenverhältnis 1:2 gezeichnet und Teile des entstehenden Möndchens quadriert. In prop. 29 soll ein Kreis zu einem flächengleichen Kreisring gemacht werden, dessen einer Grenzkreis bekannt ist. In prop. 30 wird ein regelmässiges Vieleck in ein flächengleiches Horn verwandelt, das entsteht, indem man von einem Kreissegment (mit hohlem Zentriwinkel) einen innen berührenden Kreis wegnimmt. Anschliessend verwandelt LIONNE einen gegebenen Kreis in ein flächengleiches Kreisviereck, das in prop. 31 aus Halb-und Viertelkreisbögen, in prop. 32 aus Halb-und Sechstelkreisbögen besteht. Nun wird mittels eines Halb-und eines Sechstelkreisbogens eine gemischtlinige Figur hergestellt, die gleich einem gleichseitigen Dreieck ist (prop. 33), was zu mehreren Zusätzen Veranlassung gibt (prop. 34-36).

Wird um ein Quadrat der Umkreis und über jeder Seite nach aussen der Halbkreis gelegt, so ist die Summe der Halbkreise gleich dem Umkreis. Wird das Quadrat ersetzt durch ein gleichseitiges Dreieck, so ist die Summe der Halbkreise grösser (prop. 37). Tritt aber anstelle des Quadrats ein regelmässiges Vieleck mit wachsender Eckenzahl, so wird die Summe der Halbkreise kleiner als der Umkreis und nimmt mit wachsender Eckenzahl ab (prop. 38). Diese Summe der Halbkreise lässt sich zusammenfassen zu einem einzigen Kreis. Wird dieser aus dem Umkreis herausgenommen, so ist die Fläche zwischen den Halbkreisen und dem herausgenommenen Kreis gleich der des Vielecks (prop. 39).

Nun gibt LIONNE einige Flächensätze über Vollellipsen, die für uns mittels der Inhaltsformel $ab\pi$ und algebraischer Umformungen selbstverständlich sind (prop. 40-42). Zum Schluss untersucht er die Hornfigur zwischen zwei sich von innen berührenden Kreisen (prop. 43-53), ohne dass es zu besonders überraschenden Ergebnissen kommt.

[16] So ist er durch den völlig aus der Luft gegriffenen Zusatz: *C'est ceci la quadrature des Lunes d'Hippocrate de Scio* (!) zum Vater jener schier unausrottbaren Auffassung vieler Lehrbuchschreiber geworden, die immer wieder behaupten, schon HIPPOKRATES habe sich mit den Möndchen am allgemeinen rechtwinkligen Dreieck befasst. Das trifft in Wahrheit gar nicht zu. Der richtige Sachverhalt war aus LIONNE, der hierin dem CLAVIUS[3] folgt, ganz deutlich zu entnehmen. Ueber das weitere Auftreten des Satzes am allgemeinen rechtwinkligen Dreieck in den elementaren Lehrbüchern der folgenden Zeit vergleiche man etwa J. TROPFKE: *Geschichte der Elementarmathematik* IV, 2. Auflage Berlin 1923, p. 145-46.

Hübsch ist prop. 52 (Abb. 4): ***Wenn über dem Radius BA des Kreises AEF als Durchmesser der Kreis ACB beschrieben wird, dann schneidet ein beliebiger Radius des grösseren Kreises vom Möndchen zwischen beiden Kreisen ein gemischtliniges Dreieck AEC, ab, das gleich dem Segment AIC des kleineren Kreises ist.*** Dabei sind die Bögen *AE* und *AC* gleich (prop. 53). Auf diese Weise kann also jedes Halbsegment durch einen Kreisbogen halbiert werden.

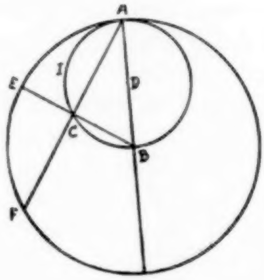


Fig. 4

6. *Schluss.* Wir fassen das Ergebnis unserer Untersuchung zusammen: LIONNE erweist sich als ein witziger Geometer. Er hat als erster eine Reihe zierlicher Quadraturen an Kreisbogenfiguren elementar ausgeführt. Seine Schrift wurde zwar rasch vergessen, aber von einigen späteren Benutzern, die sich seine Ergebnisse aneignet-en, ohne Namensnennung ausgeplündert. Sie ist also die Quelle dieser späteren Untersuchungen geblieben und daher bedeutungsvoll. Ob LIONNE für seine elementaren Quadraturen aus noch früheren Quellen schöpfen konnte oder gar geschöpft hat, kann augenblicklich nicht entschieden werden. Ergänzend füge ich einige biographische Notizen über LIONNE an. Sein Vater SÉBASTIEN war königlicher Rat, Generalsteuereinnehmer und Präsident des Rechnungshofes zu Chambéry. Unser ARTUS wurde am 1. September 1583 geboren. Er. studierte die Rechte und erwarb sich die juristische Doktorwürde. 1605 wurde er zu Grenoble Parlamentsrat und heiratete kurz darauf die erst 14—jährige ISABEAU DE SERVIEN,[17] die schon 1612 starb. ARTUS liess seinem 1611 geborenen Sohn HUGUES[18] eine

[17] Sie war die Schwester des bekannten Staatsmannes ABEL DE SERVIEN (1593-1659), der 1648 Frankreich in Münster bei Abschluss des Westfälischen Friedens vertrat und 1653 zusammen mit dem berüchtigten NICOLAS FOUQUET (1615-1680) Oberintendant der Steuern wurde.

[18] HUGUES DE LIONNE (1611-1671) wurde von seinem Oheim SERVIEN in die Staatskunst eingeführt. Von Kardinal MAZARIN wurde er besonders bevorzugt und mit wichtigen diplomatischen Missionen betraut, die er mit überragender Sach- und Menschenkenntnis auszuführen wusste. Nach MAZARINS Tod (1661) war er der verantwortliche Leiter der Aussenpolitik LUDWIG XIV.

sorgfältige Erziehung angedeihen. Der Schmerz über den allzufrühen Tod seiner Gattin verleidete ihm aber seine Amstätigkeit. Er liess sich zum Priester weihen und wurde Canonicus bei Nôtre-Dame zu Grenoble. 1634 bestellte ihn LUDWIG XIII zum Koadjutor des Bischofs von Gap und veranlasste, dass ARTUS 1637 selbst Bischof von Gap wurde. LIONNE widmete sich mit ganzer Kraft seiner Diözese. Unter anderm baute er die 1577 von den Hugenotten zerstörte Kirche wieder auf. Inzwischen stieg sein Sohn HUGUES zu immer höheren Staatsämtern empor und spielte seit 1653 als MARQUIS VON BERNY am Hofe LUGWIG XIV. eine wichtige Rolle. Ihm widmete LÉOTAUD sein "examen circuli quadraturae".[1] Man wünschte dem einflussreichen Sohn gefällig zu sein, indem man dem greisen Vater zuerst das Erzbistum von Embrun (1658), dann das reiche Bistum von Bayeux (1659) anbot, aber ARTUS wollte sich von seiner treuen Gemeinde nicht trennen und schlug beides aus. 1661 legte er die Bischofswürde freiwillig nieder und zog sich nach Paris zurück, um seninem Sohn näher zu sein. Dort starb er am 18. Mai 1663, kurz nachdem HUGUES von LUDWIG XIV. zum ersten Minister berufen worden war.

The Teachers' Department

Edited by
JOSEPH SEIDLIN and JAMES McGIFFERT

Introduction to Heaviside's Calculus

By WM. C. JOHNSON, Jr.
Rensselaer Polytechnic Institute

The simplicity which attends the solution of engineering differential equations by the use of Heaviside's operational calculus was first forcibly demonstrated to me while teaching in the Advanced Course in Engineering of the General Electric Co. at Schenectady, N. Y. There, when a problem involving ordinary differential equations was given to the students and they were supposed to use the classical method of solution, it was necessary to go far to find a problem that wouldn't be solved, and much more simply solved, by Heaviside's method. Partial differential equations were also handled effectively by this operational method, but no idea of the great saving of effort in the field can be given in this short article.

This paper is intended to serve as an introduction, and to show how the subject can be presented early in engineering mathematics, perhaps some day in freshman courses. The material, in substance, was used in the Electrical Course of the aforementioned company, in what corresponded to senior work in college. The section of this article connecting Heaviside's operators with functions of a complex variable is not necessary to an elementary understanding, but will be of great interest to any mathematician who has not seen this connection brought out.

Dean Bush* expresses this dependence very strongly when he states, "It is hardly possible properly to grasp the Heaviside operational method without some knowledge of the theory of functions of a complex variable. In fact, there is so close an interdependence that any attempt to extend the method or to use it in unusual cases will be decidedly hampered unless this theory is freely available.

*Bush, Vanneaur, *Operational Circuit Analysis*, p. 148. John Wiley & Sons, Inc., New York, 1929.

"On the other hand, even a rudimentary knowledge of the theory [of functions of a complex variable] is sufficient for some purposes. An engineer with such a knowledge can safely proceed to an extent that involves a basic understanding rather than a simple parrot-like use of formulas, and can depend upon the theory of functions to raise a danger signal in doubtful cases."

I can well remember the relief felt in finding that the operational method was firmly based on something as solid-sounding as functions of a complex variable, although at the time I knew very little about them.

COMPARISON WITH OTHER METHODS

Something must be given up to obtain the simplicity that Heaviside's method offers, but in teaching it and in using it this loss will be little felt. The operational method inherently considers a system initially at rest and it is this definiteness which makes it possible to come by simple algebraic manipulation directly to the solution with all the constants of integration belonging thereto. The healthful exercise of fitting constants to terminal conditions may in this way have been missed, but the operational method by its ease and simplicity leads to useful conceptual thinking seldom attained by use of the longer classical method.

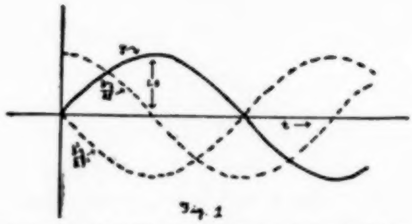
Soon in the engineering student's mind, for example, the operator p takes its place beside the very useful operator iw, as introduced by Steinmetz and others for sinusoidal-wave relations. Differential equations as a study always have seemed to the practical engineering student like judicious guessing, and anything that will nail them down and make them definite will appeal to him and make him readier to go on to higher branches of mathematics.

SINUSOIDAL WAVES

In this development for use with beginners in differential equations no excuses will be made for its being elementary. This will be its merit, if any. In the preliminary part there will be shown a few of the relations among derivatives, infinite series, exponentials, and trigonometry which every engineering sophomore is supposed to know, but with which he is seldom familiar.

Let us propose the problem of a weight bobbing up and down when suspended from a spring, to find the equation of its motion by simple mathematical reasoning and without using forces. If we pass a paper horizontally by this weight as it oscillates and have the weight

make a continuous mark on the paper we get a graph of wave motion. Using the physical idea that velocities and accelerations will yield similar curves, we can derive an equation for such a curve (Fig. 1). The student must appreciate, of course, that velocity is the first derivative or slope at any point and that acceleration is the second derivative.



Sinusoid with First and Second Derivatives.

To obtain the curve as an infinite series we write a general equation with numerical constants ***a, b, c, d***, etc.:

$$y=a+bt+ct^2+dt^3+et^4+\text{etc.}$$

$$\frac{dy}{dt}=0+b+2ct+3dt^2+4et^3+\text{etc.}$$

$$\frac{d^2y}{dt^2}=0+0+2c+6dt+12et^2+\text{etc.},$$

and so on for higher derivatives.

Without trying to be general it is very easy to find an example of an equation for y. Strangely enough we need consider the system of curves at one point only, and the easiest is the origin. For simplicity also let us assume curves for the derivatives (velocity, acceleration, and higher derivatives) like the curve of displacement and of unit height, but shifted along the time axis as shown in Fig. 1. Analysis of slopes at various points makes this seem reasonable.

Comparing equations and curves for $t=0$, we immediately identify a as zero, b as unity, c as zero, and so on. We have, then,

$$y=t-\frac{t^3}{3\cdot 2}+\frac{t^5}{5\cdot 4\cdot 3\cdot 2}-\text{etc.}$$

By now it may be suspected that this is the equation:

$$y=\sin t.$$

If the student has a very sceptical nature he can be advised to take his trigonometric tables and compare values calculated from the series with values taken from the tables, using radians for the angles, and these of small size.

Another very important point should be made at this stage. As shown in Fig. 1 the second derivative is exactly the reflection of the curve itself, and therefore the negative of it. This can also be seen from the equation in series form. We see, then, that we may write

$$\frac{d^2y}{dt^2} = -y.$$

Therefore $y = \sin t$ is a solution of this, our first differential equation. No great stretch of reasoning is needed to show that cosine t (the cosine being the derivative of the sine) also is a solution of this differential equation; as will be the sum of sine and cosine each multiplied by an arbitrary constant.

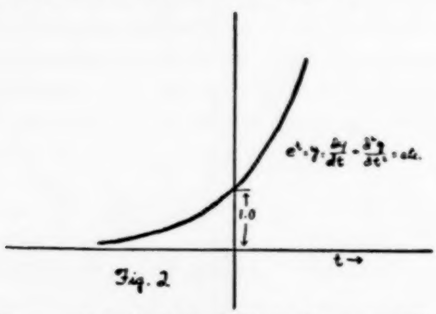
EXPONENTIALS

Now that a differential equation of second order has been solved the question arises how to solve equations of first order. The same sort of reasoning can be applied in this case if we start to find a curve for which all derivatives are alike and the equation of such a curve.

Again we need analyze the situation only at the one point $t=0$. The simplest thing turns out to be to take y and all its derivatives as unity at this point. If we then find the numerical constants for the general infinite series, we have

$$y = 1 + t + \frac{t^2}{2} + \frac{t^3}{3 \cdot 2} + \frac{t^4}{4 \cdot 3 \cdot 2} + \text{etc.} = f(t).$$

The curve for this equation is shown in Fig. 2.



Exponential of which all Derivatives are Alike.

It is rather difficult to calculate many points from the infinite series and we wish to consider finding an equation simpler in form. Perhaps the easiest way to find this is to take two infinite series of this type, multiply one by the other (for example $f(t)$ times $f(u)$), and show that $f(t+u)$ is obtained as follows:

$$
\begin{aligned}
y &= 1+t+\frac{t^2}{2}+\frac{t^3}{3\cdot 2}+\text{etc.}\\
uy &= \quad u+ut+\frac{ut^2}{2}+\text{etc.}\\
\frac{u^2y}{2} &= \quad \frac{u^2}{2}+\frac{u^2t}{2}+\text{etc.}\\
\frac{u^3y}{3\cdot 2} &= \quad \frac{u^3}{3\cdot 2}+\text{etc.}
\end{aligned}
$$

and so on.

$$f(t)\cdot f(u)=f(t+u)=1+(t+u)$$

$$+\frac{(t+u)^2}{2}+\frac{(t+u)^3}{3\cdot 2}+\text{etc.}$$

The function for which this relation holds is an exponential and the base is easiest taken as $f(1)$, given the symbol e. This is the much used Napierian base, with the numerical value 2.718...... We are now able to write the equation we have been seeking in the form

$$y=e^t.$$

This exponential function and its product by any numerical constant will give us solutions for the differential equations

$$y=\frac{dy}{dt}=\frac{d^2y}{dt^2}=\text{etc.}$$

SINUSOID AND EXPONENTIAL

It appears that we have learned something about the characteristics of e^t and *sine t*, but we seem as yet to have two separate individualities. In reality it is more a case of dual personality, and the con-

necting link is found in comparing infinite series. However, to show this relation we need to introduce another tool, $\sqrt{-1}$ or i. Consider

$$y=e^{it}.$$

Let us expand this and collect terms, as follows:

$$y=\left[1-\frac{t^2}{2}+\frac{t^4}{4\cdot 3\cdot 2}-\text{etc.}\right]+i\left[t-\frac{t^3}{3\cdot 2}-\frac{t^5}{5\cdot 4\cdot 3\cdot 2}-\text{etc.}\right]$$

$$y=\text{cosine } t+i \text{ sine } t=e^{it}.$$

Using this equation and the similar one for e^{-it}, we may solve for ***sine** t* and ***cosine** t* in terms of the exponentials. We then have equations which can be used to prove the trigonometric relations for sines and cosines of multiple angles in terms of powers of sines and cosines of single angles.

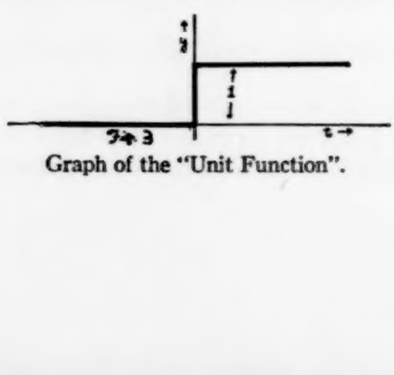
If the mathematical ideas developed in these last sections are really understood by a student he need have no trouble with engineering differential equations, but if he doesn't understand them all he will at times be lost. It is not to be inferred, of course, that he must think of them as they have been developed here.

THE HEAVISIDE OPERATION

The beauty of the Heaviside operation is that it takes the guessing out of solving the differential equations found in engineering problems. With classical methods as we have indicated, there is not only guessing at functions, but there is also speculation, if not guessing, as to how the numerical constants of integration are to be determined.

Heaviside's method gives both the function and the constants, with the assumption that the system starts from rest. Various expedients with this operational procedure can be used if the system does not start from rest, and usually the mental effort is still much less than with the older methods.

To provide that systems shall be initially at rest Heaviside introduced the "unit function," a multiplier that has zero value before $t=0$ and unit value afterwards, as shown in Fig. 3. We shall assume



Graph of the "Unit Function".

this multiplier in our equations from this point on, without writing it in as is usually done. With this understanding we have:

$$\int t^n dt = \int_0^t t^n dt = \frac{t^{n+1}}{n+1}.$$

$$\int \text{cosine } t\, dt = \int_0^t \text{cosine } t\, dt = \text{sine } t.$$

$$\int \text{sine } t\, dt = \int_0^t \text{sine } t\, dt = 1 - \text{cosine } t.$$

The operator p is used for d/dt and as such it obeys almost all of the algebraic rules. The reciprocal of p denotes integration, and with zero for the lower limit as above.

$$py = \frac{dy}{dt},$$

$$p^2y = \frac{d^2y}{dt^2}, \text{ etc.}$$

Let us consider a definite problem, for example that of the weight and spring, with the following equation:

$$\frac{d^2y}{dt^2} + y = p^2y + y = F.$$

The second derivative represents a force of acceleration; y itself represents a spring force.

This differs from the form $p^2y = -y$, which we have considered before, by the addition of the constant F. This will be a suddenly applied force needed to set the system in motion, since, as explained previously, the Heaviside method assumes a system originally at rest.

For an answer we solve this equation for y:

$$y = \frac{F}{p^2+1} = p^{-2} \cdot \frac{F}{1+p^{-2}},$$

and perform the second step of dividing numerator and denominator by p^2 in order to get descending powers of p and hence integration in

our final steps. The next step is simply to perform the division indicated by the fraction and obtain an infinite series,

$$y=p^{-2}F\left[1-\frac{1}{p^2}+\frac{1}{p^4}-\text{etc.}\right].$$

In this we perform the integrations indicated, using as limits o and t,

$$y=p^{-2}F\left[1-\frac{t^2}{2}+\frac{t^4}{4\cdot3\cdot2}-\text{etc.}\right].$$

Now, taking notice of the remaining p's, we have

$$p^2y=\frac{d^2y}{dt^2}=F\text{ cosine }t,$$

$$\frac{dy}{dt}=F\text{ sine }t,$$

$$y=F(1-\text{cosine }t).$$

Here *1 − cosine t* is the only solution that can be obtained from the conditions of the problem, whereas before we obtained the solution of the analogous form as the sum of sine and cosine terms with as yet undetermined constant coefficients.

The type of differential equation here solved is usually modified with constants multiplying the acceleration and displacement terms, but it only requires a little more algebra for solution of such a differential equation. This type of analysis applies directly to electrical circuits, where electric charge is analogous to displacement, current is analogous to velocity, and voltage is analogous to force.

EXPANSION THEOREM

With the operational method let us solve an equation of first order, introducing a numerical constant α with the displacement term,

$$\frac{dy}{dt}+\alpha y=py+\alpha y=F,$$

$$y=\frac{F}{p+\alpha}=p^{-1}\cdot\frac{F}{1+\alpha p^{-1}}.$$

If we perform the division indicated by the last fraction we obtain,

$$y=p^{-1}F\left[1-\frac{\alpha}{p}+\frac{\alpha^2}{p^2}-\frac{\alpha^3}{p^3}+\text{etc.}\right].$$

$$\frac{dy}{dt}=py=F\left[1-\alpha t+\frac{\alpha^2t^2}{2}-\frac{\alpha^3t^3}{3\cdot 2}+\text{etc.}\right]=Fe^{-\alpha t}$$

$$y=\frac{F}{\alpha}(1-e^{-\alpha t}).$$

Here again our solution is definite and contains no unknown constants of integration.

The Heaviside expansion theorem is based on a combination of this last operation with the theory of partial fractions. This is possible because of the fact that the ordinary engineering differential equations come out in the following form:

$$y=\frac{A}{(p+\alpha)(p+\beta)(p+\gamma)\cdots\cdots},$$

where A contains p to any of various powers.

By the rules for expansion of partial fractions this may be expanded into the form,

$$y=U+\frac{V}{p+\alpha}+\frac{W}{p+\beta}+\frac{X}{p+\gamma}+\text{etc.}$$

and the full solution is the sum of the solutions of these simple separate terms. A formula called the Heaviside expansion formula has been developed to combine the several necessary steps into one expression. In using this the hardest part of the solution of engineering differential equations lies in finding the roots $-\alpha$, $-\beta$, $-\gamma$, etc.

One final thing should be mentioned which makes this method more general than might at first be thought. If we are not dealing with a case where a single-step force is applied at $t=0$ (i. e. where the force times the "unit function" is applied), but there is applied instead a force of arbitrary shape, Heaviside's operational method can still be used. The scheme is simply to find the effect of applying the "unit function" and then to assume with a straight-forward integrable formula that the actual applied force is made up of an infinite number of such small sudden increments as in Fig. 4.

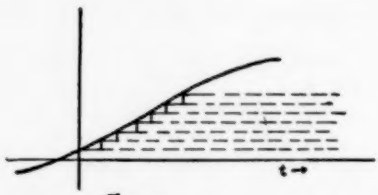


Fig. 4

Approximation by Successive "Unit Functions".

CONNECTION WITH THE COMPLEX VARIABLE

For those who are at all familiar with the theory of functions of a complex variable it should be interesting to show what form Heaviside's operators take for that theory.

It has been shown* that Heaviside's "unit function," 1, can be defined as

$$1 = \frac{1}{2\pi i}\int \frac{e^{pt}}{p}\,dp,$$

where this calls for integration in the complex plane, p being taken now as the complex variable $x+iy$.

By expansion we find that the expression

$$\frac{e^{pt}}{p} = \frac{1}{p} + t + \frac{pt^2}{2} + \text{etc.}$$

has a simple pole (root in the denominator) at $p=0$. For t negative we set up the convention of integrating to the right of the pole, getting nothing for our pains. For t positive we integrate to include the pole as in Fig. 5 and, no matter what the value of t, we get a residue of

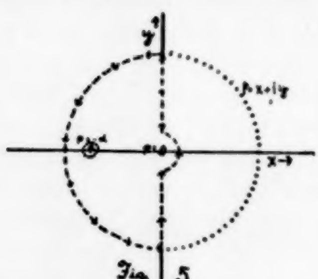


Fig. 5

Path of Integration in Complex Plane.

*Ibid., p. 156.

$2\pi i$ to cancel the other $2\pi i$. Therefore, with these conventions our "unit function" is zero before $t=0$ and unity thereafter, as it should be.

This seems arbitrary and somewhat mystifying at first, but an actual example to include another pole may clear up the application of this expression for the "unit function" as used in the complex plane.

Let us take the differential equation used before

$$py+\alpha y=F1$$

$$y=\frac{F1}{p+\alpha}$$

The form for integration in the complex plane is

$$y=\frac{1}{2\pi i}\int\frac{Fe^{pt}}{p(p+\alpha)}\,dp$$

The added pole at $p=-\alpha$ lies in the left half of the complex plane (α is a positive constant), as is almost always the case for actual systems, and integrating around both poles for t positive gives the following result, by the residue theorem:

$$y=\left[\frac{pFe^{pt}}{p(p+\alpha)}\right]_{p=0}+\left[\frac{(p+\alpha)Fe^{pt}}{p(p+\alpha)}\right]_{p=-\alpha}$$

$$y=\frac{F}{\alpha}+\frac{Fe^{-\alpha t}}{-\alpha}$$

$$y=\frac{F}{\alpha}(1-e^{-\alpha t}).$$

This is the equation obtained before by simple expansion and integration. The use of the theory of the complex variable has here seemingly only complicated matters, but as indicated in the first part of the article there are many questions in the use of Heaviside's operational calculus that can only be settled by appealing to such an integration in the complex plane.

For the benefit of those who may have been introduced to this operational procedure for the first time, it may be well to repeat that good practical use of it can be made by one without his ever having gone through the point of attack of complex-variable theory.

Mathematical World News

Edited by
L. J. ADAMS

Professor Archibald Henderson, head of the mathematics department of the University of North Carolina, reports the addition of Dr. Reinhold Baer (Göttingen) and Dr. Nathan Jacobson (Princeton) to his staff. Dr. Baer's special interests are in algebraic field and group theory, while Dr. Jacobson's are in topology. Some of Professor Henderson's recent addresses have been: ***The Celebration of the Two Hundred and Fiftieth Anniversary of Newton's Principia*** and ***Newton and Einstein—A Battle of Giants***, before the students of Limestone College, Gaffney, South Carolina; and ***The Theory of Relativity and What It Means to Us***, before the League for Political Education, Town Hall, New York City. In the forthcoming issue of the ***Journal*** of the Elisha Mitchell Scientific Society will appear a monograph by Dr. Henderson entitled ***A Classic Problem in Euclidean Geometry***, which will be dedicated to Dr. George D. Birkhoff, Harvard University.

The new volume of the ***Bulletin*** of the American Mathematical Society will be dedicated to Dr. E. R. Hedrick, who served as its editor for eighteen years.

The November, 1937 supplement of the ***American Mathematical Monthly*** is a separate pamphlet containing the names, addresses and official positions of the members of the Mathematical Association of America.

Scripta Mathematica announces the publication of portfolio II, ***Portraits of Eminent Mathematicians***, with brief biographical sketches by David Eugene Smith, professor emeritus at Columbia University. This second portfolio will include the portraits of Euclid, Cardan, Kepler, Fermat, Laplace, Hamilton, Cauchy, Euler, Cayley, Jacobi, Pascal, Chebycheff, and Poincare.

Columbia University will offer the following graduate courses in mathematics in summer session, 1938: ***Fundamental Concepts of Mathematics Curves and Families of Curves*** by Professor E. Kasner, ***Theory of Functions of a Complex Variable*** by Professor W. Benjamin Fite, ***Differential Equations*** by Professor B. O. Koopman, and ***Introduction to Higher Algebra*** by Professor Paul Smith.

Professor James McGiffert has reported that in September last, three men were elevated from instructors to assistant professors in Rensselaer Polytechnic Institute. They were Dr. Lynn Merrill, who received his B. S. and M. S. in Clarkson Tech., and his Ph.D. in Rensselaer; Dr. Ralph Huston, who obtained his B. S. in the University of Chicago, went to the University of Oxford as a Rhodes scholar and obtained a B. A. there, afterwards securing his Ph.D. in the University of Chicago; Dr. Dennis K. Ames, who received his B. A. and M. A. in Bishop's University in Lennoxville, Quebec and his Ph.D. in Yale.

Two of the most recent numbers of Mémorial des Sciences Mathématiques, published under the auspices of The Academy of Sciences of Paris, are: *Sur une conception nouvelle des forces intérieures dans un fluide en mouvement,* by M. Stanislas Zaremba, and *Mécanique analytique et mécanique ondulatoire* by M. Gustave Juvet.

The Faculty of Science, University of Geneva, Switzerland, organizes international conferences on various branches of mathematics. A special committee headed by Professor R. Wavre, selects and invites lecturers. In 1937, a conference held October 11-15 was devoted to the theory of probability.

One of the important mathematical journals devoted exclusively to research is Annali Matematica, Pura ed Applicata, Gia divetti da Francesco Brioschi, Nicola Zanichelli, Editore, Bologna. The editors include G. Fubini, T. Levi-Civita, B. Segre, F. Severi.

The annual meeting of the National Council of Teachers of Mathematics will be held at the Hotel Tryamore, Atlantic City, New Jersey on February 25 and 26, 1938.

The Twelfth Yearbook of the National Council of Teachers of Mathematics is on Approximate Computation by A. Bakst. It is published by the Bureau of Publications, Teachers College, Columbia University.

Teachers College, Columbia University, offers a field study course in Mathematics Education in Germany and England, for the summer of 1938. The date of departure is June 30th, and return from England on August 19th.

The national spring meeting of the American Society of Mechanical Engineering is scheduled for Los Angeles, California on March 23-25, 1938. The semi-annual meeting will be held in St. Louis, Mo., on June 20-24, 1938.

The meetings of the Liverpool Branch of the Mathematical Association of England are held at the University of Liverpool. At the February 28th meeting Dr. F. W. Bradley will speak on *Ideas of Elementary Mathematics Analysis for Senior Students in Secondary Schools*, and at the May 2nd meeting Prof. W. P. Milne will speak on *The Value of History of Mathematics as an Aid to Teaching.*

The directory of the junior college in the January, 1938 number of the *The Junior College Journal* lists 553 junior colleges in the United States, with a total enrollment of 136,623.

Problem Department

Edited by
ROBERT C. YATES and EMORY P. STARKE

This department solicits the proposal and solution of problems by its readers, whether subscribers or not. Problems leading to new results and opening new fields of interest are especially desired and, naturally, will be given preference over those to be found in ordinary textbooks. The contributor is asked to supply with his proposals any information that will assist the editors. It is desirable that manuscript be typewritten with double spacing. Send all communications to Robert C. Yates, College Park, Maryland.

SOLUTIONS

No. 128. Proposed by *Henry Schroeder*, Louisiana Polytechnic Institute.

What sum of money is required at 6% nominal, compounded continuously, to purchase an annuity of \$10.00 per day for twenty years? (Would like to have a formula developed by use of Differential Equations.)

Solution by *W. T. Short*, Oklahoma Baptist University.

The effective rate r equivalent to the nominal rate $j=im$, converted m times a year is given by

$$1+r=(1+j/m)^m=[(1+j/m^{m/j}]^j=e^j,^*$$

when m becomes infinite and $e=2.71828\ldots.$ In the case of a nominal rate converted continuously we see therefore that the rate is given by $r=e^j-1$, or in this case, $r=e^{.06}-1$. Therefore $1+r$ for one day would be $[e^{.06}]^{1/(365+1/4)}$. Substituting in the formula:

$$a_{\overline{n}|i}=[1-(1+i)^{-n}]/i^\dagger$$

we have as the present value:

$$\frac{10(1-e^{-1.2})}{e^{8/48703}-1}=42{,}536.$$

*See e. g. Simpson, Pirenian, Crenshaw: Mathematics of Finance, p. 182.
†Ibid, p. 202.

Note: $$e^{.06/365\frac{1}{4}} = e^{8/48700} = e^{.00016427105},$$

$$[e^{8/48700}]^{7305} = e^{1.2}, \text{ there being 7305 days in 20 years.}$$

From tables $e^{-1.2} = .301196$ and the calculation of $e^{.00016427105}$ gives 1.0001642844.

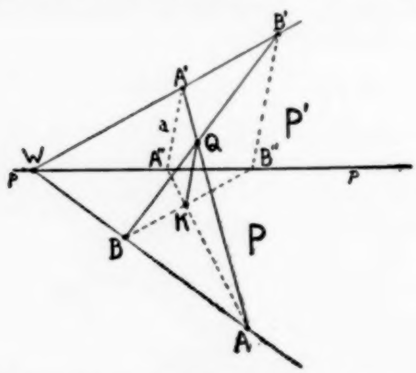
No. 159. Proposed by *T. Dantzig*, University of Maryland.

Two planes, α, β, make an angle Θ with each other. A triangle in α is in perspective from a point P with another triangle in β. Show that as Θ varies P describes a circle.

Solution by *George Dantzig*, University of Michigan.

Let A, B, C and A', B', C' be the vertices of the two traingles which lie respectively in the planes P and P'; let the intersections of the lines BC and $B'C'$, CA and $C'A'$, AB and $A'B'$ be the points U, V, W, respectively; and let the intersections of the planes P and P' be the line p. The points U, V, W must lie on the line of intersection of the planes, hence p is the axis of perspective of the two triangles. Let Q be the center of perspective of the two triangles. If we keep the plane P fixed and turn the plane P' about the line p, the axis of perspective of the two triangles remains fixed but the center of perspective Q, however, will change. We will show that the locus of Q is a circle whose plane is perpendicular to p.

In this rotation the points A', B', C' will describe circles about p. Let the centers of these circles be A'', B'', C'' and their radii be a,b,c, respectively. Call the intersections of the lines AA'' and BB'' the



point K; we shall show that the line QK is of constant length and is perpendicular to the line p. Consider the configuration after revolution through any fixed angle. In the planes $AA'A''$ and $BB'B''$ the

lines $A'A''$ and $B'B''$ are parallel and both are perpendicular to the line p. Hence the intersection of these two planes, i. e. QK, is parallel to $A'A''$ and $B'B''$ and perpendicular to p. Therefore, triangles $AA'A''$ and QKA are similar and we have the proportion $A'A''/QK = AA''/KA$. But AA'', KA, and $A'A'' = a$ are fixed lengths independent of the angle of rotation. We conclude therefore that QK is also constant. Moreover QK is perpendicular to the line p; thus the point Q describes a circle with K as center.

No. 171. Proposed by *Walter B. Clarke*, San Jose, California.

Considering only plane, obtuse, scalene triangles with sides and areas integral:

(1) What are the sides of the one of least area?

(2) What are the sides of the smallest two triangles having equal areas but all six sides different?

(3) Of those having areas equal numerically to perimeters, what is the one of least area?

(4) What is a value of x such that two triangles of equal area have their semi-perimeters equal to x^2+x and x^2-x, respectively?

Solution by *E. P. Starke*, Rutgers University.

Temporarily disregarding the scalene-obtuse requirement, the problem may be reworded; find three numbers ($s-a$, $s-b$ and $s-c$ in the customary notation) such that their sum times their product is a perfect square. (If a, b and c and the area K are integers, then s must be an integer, since the equation $s(s-a)(s-b)(s-c) = K^2$ has integral coefficients.) Let arbitrary values be assigned to $(s-a)$ and $(s-b)$ with the sum c and the product p. We must then determine $x = s-c$ such that

$$xp(x+c) = K^2 \tag{1}$$

is a perfect square. This is easy to do by trial, at least for all the desired information except (4), since K need not be greater than 40. Examining all the possibilities with $K < 40$ and removing all which represent right- or acute-angled or isosceles triangles, we find for (1) 4, 13 and 15 as the sides of the triangle whose area 24 is the smallest; for (2) 3, 25, 26 and 9, 10, 17 are the sides of two triangles with area 36; for (3) the last named triangle has area and perimeter numerically equal.

The Diophantine equation (1) suggests more extended treatment, particularly to obtain an answer for (4). If (x, K) is a pair of integers satisfying (1), so also is (x', K') given by

$$x' = \alpha x + \beta K + x_0$$

$$K' = p\beta x + \alpha K + K_0$$

where α, β are the smallest* values for which $\alpha^2 = p\beta^2 + 1$, and $x_0 = c(\alpha - 1)/2$, $K_0 = \beta c p/2$. Thus $px_0(x_0 + c) = K_0^2$. Since (0,0) is an obvious solution of (1), we may obtain an infinite sequence of pairs of values which also satisfy (1). In certain cases there are other pairs between (0,0) and (x_0, K_0) which satisfy (1). There are then two or more distinct sequences of solutions. All solutions of (1) are thus determined by finding by trial all solutions of (1) with $0 < x < x_0$, and applying the reduction formulas to each. Certain simplifications are obvious: if p contains a square factor, it may be removed and a smaller value employed for K; if g is a factor of c such that p is a quadratic non-residue of g, then x and K must be multiples of g, so that the factor g can be suppressed.

By treating as above all triangles for which $K < 500$, we have an answer to (4); the two triangles whose sides are 21, 85, 104 and 15, 106, 119 respectively have area 420 each and perimeters $x^2 - x$ and $x^2 + x$ where $x = 15$. A number of other curiosities appear from the computations thus indicated: (a) two different triangles having equal areas and equal perimeters (the sides are 21, 41, 50 and 26, 35, 51, respectively; (b) three distinct triangles having equal areas and such that one of them has two sides in common with each of the others (26, 73, 97 and 26, 51, 73 and 26, 35, 51); (c) K is always a multiple of 6 (consider equation (1) with respect to the moduli 3 and 4).

Of interest is a different approach, easier than the above except that it is difficult to determine all solutions having a given maximum for K. If a, b, c and K are rational, then $\tan(A/2) = K/s(s-a)$ is rational. On the other hand if two half-angles have rational tangents so has the third; then $s(s-a)/K$ and $s(s-b)/K$ are rational and their sum sc/K is rational, with analogous statements for sb/K and sa/K; thus the ratios of the sides are rational. If an addition one side is rational, then all sides are rational and K is rational.

Hence let $\tan(A/2) = m/n$ and $\tan(B/2) = p/q$, where we may require the integers m, n, q, p to satisfy $q/p > m/n > 1$ so that $A > 90°$ and $A + B < 180°$. Then $\tan(C/2) = (nq - mp)/(mq + pn)$ and

*If the values of x_0, K_0, come out fractional, we use the next pair of solutions of $\alpha^2 = p\beta^2 + 1$; or we might compute the whole sequence suggested above and discard the alternate pairs.

$a:b:c:s = mn(p^2+q^2) : pq(m^2+n^2) : (np+mq)(nq-mp) : nq(mq+pn)$. If a, b, c and s be set equal to these four integers, then

$$K = mnpq(np+mq)(nq-mp).$$

Of course any integral factor introduced (or suppressed) in a, b, c and s must be squared before being introduced (or removed) from K. These formulas give all the solutions to the problem.

It thus appears that the numbers

$$np(nq-mp), \quad mq(nq-mp), \quad mp(np+mq),$$

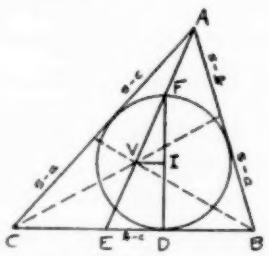
where m, n, p and q are arbitrary integers, have the property that their sum times their product is a perfect square. Conversely, if three integers have this property they are proportional to three integers given above.

No. 172. Proposed by *Walter B. Clarke*, San Jose, California.

Show that if the X-line (the line segment joining incenter to verbicenter) is parallel to one side of a triangle then the sides form an arithmetic progression, the constant difference being the length of the X-line.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

In the triangle ABC let the X-line, VI, be parallel to BC, and let DF be the diameter of the incircle perpendicular to BC at D. Then F falls on the verbisector AE (No. 121, October, 1936, p. 56). The cevians to the points of contact of the three ex-circles are concurrent at V, so $AV/VE = (s-c)/(s-a) + (s-b)/(s-a) = a/(s-a)$, and $AV/AE = a/s$. Now $AF/FE = (s-a/a$. (No. 195, solution to be published), so $FE/AE = a/s$. Hence $AF = VE$.



Now since I is the midpoint of FD and VI is parallel to BC, $VI = 1/2\,ED = 1/2(b-c)$ and

$$FV = VE = AF = 1/3AE = 1/3\sqrt{\frac{s}{a}\,[a(s-a)+(b-c)^2]}.$$

$FI=r=\Delta/s$. FIV is a right triangle, so $\overline{FI}^2+\overline{VI}^2=\overline{FV}^2$. Substituting the given values in the equation and simplifying,

$$(b+c-2a)\{a[-4a^2+3a(b+c)+(b+c)^2]+(b-c)^2(8a+2b+2c)\}=0.$$

The second factor cannot equal zero, since $(b+c)>a$, so $a={}^1\!/_2(b+c)$. That is, a is the arithmetic mean between b and c and the common difference is ${}^1\!/_2(b-c)=VI$.

No. 176. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Find all positive integral values of n which satisfy the equation:

$$(2+\sqrt{2})^n+(2-\sqrt{2})^n+2^n+2^{2n-1}=\frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-1)}{n!}\cdot 2^{n+2}.$$

Partial solution by the *Proposer*.

The seven integers from 1 to 7 satisfy the equation, and there seem to be no others which will do so.

Editor's Note. That the seven integers named are the only solutions may be shown as follows: The left member is evidently divisible by $2^{[n/2]+1}$, where $[n/2]$ indicates the greatest integer not greater than $n/2$. The right member is divisible* by 2^t but not by 2^{t+1}, where $t=n+2-[n/2]-[n/4]-[n/8]-\cdots$. If then $t<[n/2]+1$, the proposed equation is clearly impossible. Hence there is no solution when

$$n+2-[n/2]-[n/4]-[n/8]-\cdots<[n/2]+1.$$

Since $n\geqq 2[n/2]$, this inequality may be reduced to

$$1<[n/4]+[n/8]+\cdots.$$

Whence $n \ngtr 8$.

No. 179. Proposed by *V. Thébault*, Le Mans, France.

Find a perfect square of five digits such that the eight digits with which the number and its square root are written, in the system of base 8, are all different.

Solution by the *Proposer*.

If x^2 is the required square, it is not difficult to show that $8^2\leqq x<8^{5/2}$, and that $x+x^2$ is congruent (mod 7) to the sum of the digits so that $x+x^2\equiv 0$ (mod 7) or $x\equiv 0$ or 6 (mod 7). Writing down

*It is known that $n!$ is divisible by 2^s but not by 2^{s+1} where $s=[n/2]+[n/4+[n/8]+\cdots$ See Carmichael *Theory of Numbers* (1914), page 25.

the values of x which satisfy these conditions, which do not end in 0 or 1, and which present no duplication of digits, we find twenty values to test by multiplication. $256^2=73104$ thus appears as the unique solution.

No. 181. Proposed by *A. Gloden*, Luxembourg.

Find a perfect square of nine digits of the sort:

$$abcdefghi$$

such that
$$\begin{cases} abc=u^2+4 \\ def=v^2 \\ ghi=w^2+13. \end{cases}$$

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

Let N^2 be the desired square. Then

1) N^2 has nine digits and $a\neq 0$; hence $10000\leqq N<31624$.

2) The units digit of a square number is 0, 1, 4, 5, 6 or 9; the tens digit is odd when the units digit is 6, otherwise the tens digit is even. If a square terminates with 29 or 69, the hundreds digit is odd; and if it terminates with 49 or 89, the hundreds digit is even. When the values $w=1,2,\cdots,31$ are examined and these conditions are simultaneously imposed on N^2 and w^2+13, the only possible values of ghi are seen to be 049, 209 and 689. Whence N is of one of the forms, $250k\pm 7$ or $250k\pm 47$ or $250k\pm 117$. There are thus twenty-four eligible values of the last three digits of N.

3) There are twenty-two eligible values of abc. With the aid of a table of squares, these values of abc establish limiting brackets within which $N/10$ must fall. These, together with (2), restrict the possible values of the three terminal digits of N to 117, 133, 203 and 243, and the possible values of N to thirteen. When these are actually squared the restriction which v^2 places on f determines the unique solution to be $18117^2=328225689$, where $u=18$, $v=15$, $w=26$.

Also solved by the *Proposer*.

No. 183. Proposed by *G. W. Wishard*, Norwood, Ohio.

Prove the following rule for finding whether any given number is divisible by 3: Pass your pencil over the given number, jumping every 0, 3, 6 or 9, and letting it touch each 1, 4 or 7 in one place, and

each 2, 5, 8 in two places. Count 1, 2, 3, 1, 2, 3, as the pencil touches. If the last count is 3, the number is divisible by 3. Otherwise, the last count will be the remainder.

Solution by *C. W. Trigg*, Cumnock College, Los Angeles.

Let N be the given number, let a_i represent the frequency of the digit i in N, and put r_i for the remainder obtained in dividing $a_i \cdot i$ by 3. Then $a_i \cdot i \equiv r_i \pmod 3$. For the sum S of the digits of N we thus have $S \equiv \sum a_i \cdot i \equiv \sum r_i \pmod 3$. The proposed rule adds the remainders secured when each digit is divided by 3, progressively discarding multiples of 3 as they occur, to obtain a final remainder R. Hence, evidently, $\sum r_i \equiv R \pmod 3$. But it is known that $N \equiv S \pmod 3$; thus $N \equiv R \pmod 3$, which proves the rule.

Also solved by the ***Proposer***.

PROPOSALS

No. 206. Proposed by *George Dantzig*, University of Michigan.

Consider three circles in parallel planes. A cone (in fact, two cones) can be made to pass through two of the circles. Show that by a proper selection the three vertices of the cones passing through the pairs of circles will lie on a straight line. (This is a generalization of No. 159, the solution of which appears in this issue).

No. 207. Proposed by *W. V. Parker*, Louisiana State University.

Show that $y = k \cdot x \cdot \log x$ has points of maximum and minimum curvature if $k > 2/3$. Determine all positive rational values of k such that $\log x$ is rational at these points.

No. 208. Proposed by *Albert Farnell*, Centenary College, La.

Find the hodograph of a point on the ellipse which moves so that a line joining it with one of the foci covers equal areas in equal times.

No. 209. Proposed by *C. N. Mills*, Illinois State Normal University.

The axes of three right circular cylinders, with equal radii R, are concurrent and mutually perpendicular. Show that the volume of the solid bounded by the cyclinders is $8R^3(2-\sqrt{2})$.

No. 210. Proposed by *A. Moessner*, Nurnberg-N, Germany.

What is the general solution in integers of the system:

$$2(a^v+b^v)=x+y+z+w$$

$$2(a^{2v}+b^{2v})=x^2+y^2+z^2+w^2$$

$$2(a^{3v}+b^{3v})=x^3+y^3+z^3+w^3\ ?$$

An example of such a solution is given by $v=3$, $a=2$, $b=12$, $x=-336$, $y=696$, $z=1040$, $w=2072$.

No. 211. Proposed by *E. P. Starke*, Rutgers University.

Consider the series of Fibonacci (Leonardo of Pisa); 1, 1, 2, 3, 5, 8, 13, $\cdots$, where $a_{n+1}=a_{n-1}+a_n$, and show that, for every prime p,

(1) a_p is relatively prime to all preceding terms, and
(2) there are infinitely many terms divisible by p.
(3) Also find necessary conditions on r in order that a_r shall be a prime.

No. 212. Proposed by *Walter B. Clarke*, San Jose, California.

Construct a triangle having its Nagel point on one of its sides.

No. 213. Proposed by *J. Rosenbaum*, Bloomfield, Connecticut.

Prove that the product of two numbers, each of which is of the form p^2+pq+q^2, is of this form in four ways.

No. 214. Proposed by *David Amidon*, Central High School, Newark, N. J.

On the altitude AD of triangle ABC, select an arbitrary point P. Let BP meet AC in E and CP meet AB in F. Show that (a) angle EDF is bisected by AD and (b) EF and BC meet in a fixed point however P may be chosen.

No. 215. Proposed by *V. Thébault*, Le Mans, France.

Consider a circle (0), of radius R, a chord AB whose distance from the center 0 is d and a *variable* point C upon that chord between A and B. Let (ω_1), (ω_2) be circles tangent to each other at C and tangent to (0). Draw the circle (0_1) tangent to the circles (0), (ω_1), and (ω_2); then the circles (0_2) tangent to the circles (0), (0_1) and to one or the

other of the circles (ω_1), (ω_2); etc., calling (0_n) the circles tangent to (0), (0_{n-1}) and to one of the circles (ω_1), (ω_2).

(1) Find the locus of the centers of the circles (0_1), (0_2), $\cdots (0_n)$.

(2) Show that for the circle (0_n), for example, the locus is an ellipse with focus 0 and eccentricity $1/2n(1 \pm nd/2R)$.

No. 216. Proposed by *V. Thébault*, Le Mans, France.

What are the last three digits of the number 7^{9999}?

Reviews and Abstracts

Edited by
P. K. SMITH and H. A. SIMMONS

Scripta Mathematica Forum Lectures. By C. J. Keyser, D. E. Smith, E. Kasner and W. Rautenstrauch. The Scripta Mathematica Library, number 3. Published by Yeshiva College, New York City. Ninety-four pages.

This is a series of popular lectures by well known Columbia University professors, who are recognized as being fitted for the accomplishment of the end in view. This book should help stem the tide of public opinion which seems to have been running so strongly against mathematics and things mathematical.

In the first lecture Professor C. J. Keyser presents the World Theory of the late William Benjamin Smith. The reviewer is not disposed to attempt a criticism of the philosophical aspects of this lecture. He wishes to commend the readable presentation of this discussion, which is concerned largely with objectivity and subjectivity, with the psychic and the physical.

In the second lecture Professor D. E. Smith presents for the layman a running narrative of mathematics. He begins with the probable early contributions to mathematical knowledge by India, Iraq, Egypt and China. He then surveys developments and connects many of them with historical events, such as migrations, conquests and the printing press. He ends with a list of 38 "giants" who made notable contributions before 1800, such, for example, as

"—300. Euclid. Geometry, Optics, Theory of Numbers."

Dr. Smith attributes to the puzzle solving instinct the impelling force for solving mathematical problems. He says, "It is the joy of the puzzle which leads most pupils to solve a set of simultaneous equations and the lawyer to solve the legal problems which a new case proposes." This served to recall to the reviewer a statement from H. Poincaré's La Science et l'Hypothèse:

"L'esprit n'use de sa faculté créatrice que quand l'expérience lui en impose la nécessité." Any comment as to variance in the point of view we leave to the more philosophically minded reader.

The third lecture, by Professor E. Kasner, may be described as characteristic. It is stimulating, comprehensible and perhaps as com-

prehensive as the occasion would warrant. He discusses the use of some ordinary words such as, ***simple, group, family,*** when used in a mathematical sense. He also gives some words of his own invention, such as ***polygenic, turbine, parhexagon,*** with illustrations. A quarter or more of the lecture is devoted to the concept of infinity, with several examples of large finite numbers. However, one example, to which we take exception is that of a supposed mathematical expression for the avowed love of a woman in a case for divorce. Since the woman's inept hyperbole was probably indicative of her previous conduct, she was no doubt fortunate if she escaped with a simple divorce.

The fourth lecture, by Professor W. Rautenstrauch, contains a radical proposal. He draws, by analogy with the biological world, a scheme whereby our entire system of government and economics should be administered in a manner like that pertaining to our Universities. Unfortunately, he does not state whether his proposal applies to actual or ideal university administration. In the former case he should have specified the particular institutions to which he had reference. Should our national government adopt methods pursued by some of our universities, the only natural and justifiable result would be a first class revolution.

University of South Carolina J. B. COLEMAN.

(1) *Die Georg-August-Universität zu Göttingen 1737-1937.* By Götz von Selle. Vandenhoeck & Ruprecht, Göttingen, 1937. 398 pages.

(2) *Bildnisse Göttinger Professoren aus zwei Jahrhunderten.* Edited by Max Voit. Vandenhoeck & Ruprecht, Göttingen, 1937.

Its Bicentennial Jubilee which the University of Göttingen celebrated the last of June, 1937, possessed of course great objective and personal interest for many members of the Deutsche Mathematiker-Vereinigung in view of the special mathematical importance of Göttingen. The history of mathematical study there in the first century of its existence was presented in detail by Konrad Müller in his dissertation of 1904, the 19th century up to the outbreak of the World War by the reviewer in his ***Imuk**** paper of 1916. The commemorative volume which the librarian Götz von Selle wrote for the Jubilee at the request of the rector, cannot of course treat with equal completeness the cultivation of mathematical sciences in Göttingen. The author,

*International Commission on the Teaching of Mathematics.

who is himself a non-mathematician, has, however, understood very well how to depict the activity of many mathematicians, physicists and astronomers in the framework of the entire development of the university, especially the achievement of Felix Klein.

On page 327 occurs a confusion of the "Erlanger Programm" with the partially published inaugural lecture of Klein, from which the reviewer was able to quote particularly characteristic passages in his *Imuk* paper, as well as in his eulogies of Klein (*Leopoldina*, 1926; *Sitzungsberichte der Berliner Mathematischen Gesellschaft*, 1926). On page 285 the year of Gauss' death is not correct: 1855, not 1858 (correct however in the index). The *Disquisitiones arithmeticae* appeared in 1801, not 1808. When on page 305 the cultivation of music in Göttingen by Rudolph Wagner, professor of pathological anatomy, is especially praised, the same could have been reported of the professor of theoretical physics, W. Voigt, who is not named. Strangely enough the political economist Lexis (originally a mathematician) is not mentioned; as an adviser of the ministry and special confidante of Althoff he exerted a great, beneficial influence. The Pfaff† mentioned on page 103 to whom Kaestner wrote about a definition of the straight line, was not a professor in Giessen, as indicated in the index, but in Helmstedt.

Such little criticisms, which could probably be increased, should however in no wise reduce the collective value of the book. On the contrary: the reading of it offers a very rich enjoyment because it makes recognizable the general scientific, political and intellectual currents, and shows how from the beginning of the University of Göttingen, munificently endowed by the government, was able to develop its very special individuality, which united research and teaching in happy fashion. It is to be hoped that a history of the university a hundred years from now will be able to paint an equally pleasing picture, especially of the mathematical sciences.

The collection of 226 pictures edited by the anatomist Max Voit as a Jubilee publication of the alumni association and Foundation-League of the university offers a valuable supplement. The following physicists, mathematicians and astronomers are represented: Segner, Penther, Tobias Mayer, Lowitz, Kaestner, Lichtenberg, Johann Tobias Mayer, Gauss, Harding, Thibaut‡, Wilhelm Weber (after a lithograph of 1856 and a photograph in his last years), Listing, Dirichlet, Riemann, Kohlrausch, Klinkerfuss, Clebsch, Klein, Voigt, Riecke, Hilbert, Runge, H. Th. Simon, Prandtl. Unfortunately there is missing

†Cf. National Mathematics Magazine, March, 1937, p. 263.
‡Cf. National Mathematics Magazine, April, 1937, p. 318.

a picture of H. A. Schwarz§ who during the years 1875-1892 was a very well known personality in Göttingen.

Frankfort on the Main, Germany WILHELM LOREY

Projective Geometry. By Boyd C. Patterson, New York, John Wiley & Sons, 1937. xiv+276 pages. Price $3.50.

The author has designed his book to fit the needs of college students. The text calls for only a previous course in elementary geometry. The method adopted is that of synthetic geometry. The scope of the book permits the teacher to conduct either a long or a short course. Hence the book is so arranged that it permits a considerable degree of flexibility in its use. A suggested outline is given for either a 45 or 90 lessons course.

The following general topics are adequately treated: duality, perspectivity—Desargues' Theorem, harmonic sets, metric properties, double ratio, projectively related primitive forms, conics, Pascal's and Brianchon's theorems, theory of pole and polar, ruled surfaces, extended theory of projectivity, theory of involution, imaginary elements, foci and focal properties of conics, planar collineation, and metrical specializations. A wide assortment of well chosen problems is included in each chapter.

The propositions are clearly and concisely stated. Well constructed illustrative figures should prove helpful to the student in visualizing the problems. This text provides one of the most complete and teachable elementary treatments of projective geometry that has appeared recently in English.

Agnes Scott College HENRY A. ROBINSON

Introduction to Mathematics. By H. R. Cooley, David Gans, Morris Kline and H. E. Wahlert. Houghton Mifflin Company, New York, 1937. vi+634 pages.

This text does exceedingly well that which is stated on the title page, viz. "A survey emphasizing mathematical ideas and their relations to other fields of knowledge." It contains more than a year's class work, but any class using the text should be encouraged to read the entire volume. The book is so written that it is within the mental

§A photograph of Schwarz, as well as most of the others mentioned here, is in the collection of G. W. Dunnington.

reach of every freshman whether he likes or dislikes mathematics, whether his mathematical preparation has been ample or fragmentary; and the topics contained in the text are those with which every college student should be familiar.

Parts I, II, and III contain some of the elementary ideas in mathematics, knitting them together in a readily understandable form. It comes short of the orthodox first year of College Mathematics; but for the student who does not intend to pursue the subject beyond the first year in college, it furnishes an interesting and informing course. Part IV however departs somewhat from the usual freshman topics. These five topics, Infinite Classes, Non-Euclidean Geometry, The Theory of Relativity, and The Further Significance of Mathematics for Other Fields of Knowledge, will challenge the mental acumen of every thoughtful student.

It is a deligthful book, captivatingly written, and exhibits a logic that is convincing, a clarity that opens its thought to every freshman, a description of the wide range and deep seated influence of mathematics, a cogent tracing of its origins, expansions and contagion.

Rutgers University RICHARD MORRIS